A NOTE ON THE DISTRIBUTION OF THE RATIO OF SAMPLE STANDARD DEVIATIONS IN RANDOM SAMPLES OF ANY SIZE FROM A BI-VARIATE CORRELATED NORMAL POPULATION

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In this note the distribution of the ratio of sample standard deviations in random samples of size N drawn from a bi-variate correlated normal population has been obtained.

Consider a random sample $(x_1, y_1), (x_2, y_2) \cdots (x_N, y_N)$ of size N drawn from a bi-variate correlated normal population. Let

$$\bar{x} = \sum_{i=1}^{N} \frac{x_i}{N},$$

$$\bar{y} = \sum_{i=1}^{N} \frac{y_i}{N},$$

$$m_{20} = \sum_{i=1}^{N} \frac{(x_i - \bar{x})^2}{N\sigma_1^2} = \frac{S_1^2}{\sigma_1^2},$$

$$m_{11} = \sum_{i=1}^{N} \frac{(x_i - \bar{x})(y_i - \bar{y})}{N\sigma_1\sigma_2} = \frac{\gamma s_1 s_2}{\partial_1 \sigma_2}$$

and

$$m_{02} = \sum_{i=1}^{N} \frac{(y_i - \bar{y})^2}{N\sigma_2^2} = \frac{S_2^2}{{02}^2}$$

where σ_1^2, σ_2^2 are the population variances and S_1^2, S_2^2 are the sample variances of the two variables x and y respectively and γ is the product moment correlation coefficient.

Joint distribution of m_{20}, m_{11}, m_{02} has the frequency function f_N given by

$$f_N(m_{20}, m_{11}, m_{02}) = \frac{1}{4\pi}\left(\frac{N^2}{1-\rho^2}\right)^{\frac{N-1}{2}} \frac{(m_{20}\,m_{02} - m_{11}{}^2)^{\frac{N-4}{2}}}{(N-3)\,!}$$

$$\times \exp\left[-\frac{N}{2(1-\rho^2)}(m_{20} - 2\rho m_{11} + m_{02})\right]$$

in the domain $m_{20} > 0$, $m_{02} > 0$ and $m_{11}{}^2 > m_{20}m_{02}$, while $f_N = 0$ outside this domain.

Let $m_{20} = \omega\delta^t$, $m_{02} = \omega e^{-t}$, $m_{11} = \gamma\sqrt{m_{20}m_{02}} = \gamma\omega$ then under the null hypothesis that $\sigma_1{}^2 = \sigma_2{}^2$ we have $e^t = \frac{S_1}{S_2}$. After transforming the frequency function of the Joint distribution of γ, t and ω is given by

$$f_N(\gamma, t, w) = \frac{1}{2\pi}\left(\frac{N^2}{1-\rho^2}\right)^{\frac{N-1}{2}} \omega^{N-2}\frac{(1-\gamma^2)^{\frac{N-4}{2}}}{(N-3)\,!}$$

$$\times \exp\left[\frac{-N\omega}{1-\rho^2}(\cosh t - \rho\gamma)\right]$$

in the domain of $\gamma \rightarrow -1$ to $+1$, $\omega > 0$ and $t \rightarrow -\sim$ to $+\sim$ while $f_N = 0$ outside this domain.

From above the frequency function of t will be

$$f(t) = \frac{1}{2\pi}\left(\frac{N^2}{1-\rho^2}\right)^{\frac{N-1}{2}} \int_0^{\sim}\int_{-1}^{+1} e^{-\frac{N\omega}{1-\rho^2}\cosh t}\ \omega^{N-2}$$

$$e^{\frac{Nw\rho r}{1-\rho^2}}(1-\gamma^2)^{\frac{N-3}{2}-\frac{1}{2}}\ dwd\gamma$$

Watson (1944) has shown that

$$I_n(x) = \frac{1}{\sqrt{\pi}}\frac{\left(\frac{x}{2}\right)^n}{\sqrt{n+\frac{1}{2}}}\int_{-1}^{+1} e^{xr}(1-\gamma^2)^{n-\frac{1}{2}}\ dr$$

and

$$\int_0^{\sim} e^{-ax} J_n(bx)\, x^{n+1}\, dx = \frac{\sqrt{n+3/2}\ a\,(2b)^n}{(a^2+b^2)^{n+3/2}} \times \frac{2}{\sqrt{\pi}}$$

if $R(a \pm ib) > 0$, $R(2n) > 0$ it is true even if $n = 0$

Hence

$$f(t) = \frac{2^{N-3}}{\pi} (1-\rho^2)^{\frac{N-1}{2}} \frac{\sqrt{N/2}\,\sqrt{(N-2)/2}}{N-2} \times \frac{\cosh t\,dt}{(\cosh^2 t - \rho^2)^{N/2}}$$

for

for $N \geqslant 3$ and $f(t) = 0$ for $N = 1, 2$.

Let $P(t_0)$ denote the probability that t will lie in the interval $-\sim$ and t_0, then

$$P(t_0) = \frac{\int_{-\sim}^{t_0} \frac{\cosh t\,dt}{(1-\rho^2+\sinh^2 t)^{N/2}}}{\int_{\sim}^{\sim} \frac{\cosh t\,dt}{(1-\rho^2+\sinh^2 t)^{N/2}}}$$

$$= \frac{\int_{-\sim}^{Z_0} \frac{d^2}{\left(1+\frac{Z^2}{N-1}\right)^{N/2}}}{\int_{-\sim}^{+\sim} \frac{d^2}{\left(1+\frac{Z^2}{N-1}\right)^{N/2}}} = \frac{\sqrt{N/2}}{\frac{\sqrt{N-1}}{2}\sqrt{\pi\,(n-1)}} \int_{-\sim}^{Z_0} \frac{d^2 dz}{\left(1+\frac{Z^2}{N-1}\right)^{N/2}}$$

where

$$\frac{\sinh t_0}{\sqrt{1-\rho^2}} = \frac{Z_0}{\sqrt{N-1}} \quad -(1)$$

and

$$e^{t_0} = S_1/S_2.$$

The value of Z_0 in the above relation can be obtained for any value of $P(t_0)$ as follows:—

$$1 - P(t_0) = \frac{\sqrt{N/2}}{\frac{\sqrt{N-1}}{2}\sqrt{\pi\,(N-1)}} \int_{Z_0}^{\sim} \frac{dZ}{\left(1+\frac{Z^2}{N-1}\right)^{N/2}}$$

$$= \frac{P_1(Z_0)}{2}$$

where

$$P_1(Z_0) = \frac{2\sqrt{N/2}}{\frac{\sqrt{N-1}}{2}\sqrt{\pi\,(N-1)}} \int_{Z_0}^{\sim} \frac{dz}{\left(1+\frac{Z^2}{N-1}\right)^{N/2}}$$

Then

$$P_1(Z_0) = 2\,[1 - P(t_0)]$$

For any given value of $P_1(Z_0)$, the value of Z_0 can be obtained from the t-table for $(N-1)$ degrees of freedom. By substituting the value of Z_0 in (1) the value of the ratio S_1/S_2 can be obtained. These values for different values of P, N and level of significance can be tabulated.

TABLE I

$\bar{P} = \cdot 80$

N / ρ	0	·1	·2	·3	·4	·5	·6	·7	·8	·9
3	2·00	1·99	1·97	1·94	1·90	1·84	1·76	1·66	1·54	1·37
4	1·71	1·70	1·69	1·67	1·64	1·58	1·54	1·48	1·39	1·27
5	1·57	1·57	1·56	1·54	1·52	1·48	1·44	1·39	1·32	1·22
6	1·49	1·48	1·48	1·46	1·44	1·41	1·38	1·33	1·27	1·19
7	1·43	1·43	1·42	1·41	1·39	1·37	1·33	1·29	1·24	1·17
8	1·39	1·39	1·38	1·37	1·35	1·33	1·30	1·27	1·22	1·15
9	1·36	1·36	1·35	1·34	1·32	1·30	1·28	1·24	1·20	1·14
10	1·33	1·33	1·32	1·31	1·30	1·28	1·26	1·23	1·19	1·13
11	1·31	1·31	1·30	1·29	1·28	1·26	1·24	1·21	1·18	1·12
12	1·29	1·29	1·29	1·28	1·27	1·25	1·23	1·20	1·17	1·12
13	1·28	1·28	1·27	1·26	1·25	1·24	1·22	1·19	1·16	1·11
14	1·27	1·26	1·26	1·25	1·24	1·23	1·21	1·18	1·15	1·11
15	1·25	1·25	1·25	1·24	1·23	1·22	1·20	1·17	1·14	1·10
16	1·24	1·24	1·24	1·23	1·22	1·21	1·19	1·17	1·14	1·10
17	1·23	1·23	1·23	1·22	1·21	1·20	1·18	1·16	1·13	1·09
18	1·23	1·22	1·22	1·22	1·21	1·19	1·18	1·16	1·13	1·09
19	1·22	1·22	1·21	1·21	1·20	1·19	1·17	1·15	1·12	1·09
20	1·21	1·21	1·21	1·20	1·19	1·18	1·17	1·15	1·12	1·08
21	1·21	1·20	1·20	1·20	1·19	1·18	1·16	1·14	1·12	1·08
22	1·20	1·20	1·20	1·19	1·18	1·17	1·16	1·14	1·11	1·08
23	1·19	1·19	1·19	1·18	1·18	1·17	1·15	1·13	1·11	1·08
24	1·19	1·19	1·19	1·18	1·17	1·16	1·15	1·13	1·11	1·08
25	1·19	1·18	1·18	1·18	1·17	1·16	1·14	1·13	1·11	1·07
26	1·18	1·18	1·18	1·17	1·16	1·15	1·14	1·12	1·10	1·07
27	1·18	1·18	1·17	1·17	1·16	1·15	1·14	1·12	1·10	1·07
28	1·17	1·17	1·17	1·16	1·16	1·15	1·14	1·12	1·10	1·07
29	1·17	1·17	1·17	1·16	1·15	1·14	1·13	1·12	1·10	1·07
30	1·17	1·17	1·17	1·16	1·15	1·14	1·13	1·11	1·09	1·07

TABLE II

$P = \cdot 90$

N / ρ	0	·1	·2	·3	·4	·5	·6	·7	·8	·9
3	3·00	2·97	2·95	2·88	2·80	2·68	2·52	2·33	2·08	1·73
4	2·31	2·31	2·29	2·24	2·19	2·11	2·01	1·88	1·71	1·49
5	2·02	2·02	2·00	1·96	1·92	1·86	1·78	1·68	1·56	1·38
6	1·85	1·85	1·83	1·81	1·77	1·77	1·65	1·57	1·46	1·32
7	1·74	1·74	1·73	1·70	1·67	1·63	1·57	1·50	1·41	1·28
8	1·66	1·66	1·65	1·63	1·60	1·56	1·51	1·48	1·37	1·25
9	1·60	1·60	1·59	1·57	1·55	1·51	1·47	1·41	1·33	1·23
10	1·56	1·55	1·54	1·53	1·50	1·47	1·43	1·38	1·31	1·22
11	1·52	1·52	1·51	1·49	1·47	1·44	1·40	1·35	1·29	1·20
12	1·49	1·48	1·48	1·46	1·44	1·41	1·38	1·33	1·27	1·19
13	1·46	1·46	1·45	1·44	1·42	1·39	1·35	1·31	1·26	1·18
14	1·44	1·43	1·43	1·41	1·40	1·37	1·34	1·30	1·24	1·17
15	1·42	1·41	1·41	1·40	1·38	1·35	1·33	1·28	1·23	1·16
16	1·40	1·40	1·39	1·38	1·36	1·34	1·31	1·27	1·22	1·16
17	1·38	1·38	1·37	1·36	1·35	1·33	1·30	1·26	1·22	1·15
18	1·37	1·37	1·36	1·35	1·33	1·31	1·29	1·25	1·21	1·150
19	1·36	1·35	1·35	1·34	1·32	1·30	1·28	1·24	1·20	1·14
20	1·35	1·34	1·34	1·33	1·31	1·29	1·27	1·24	1·19	1·14
21	1·33	1·33	1·33	1·32	1·30	1·29	1·26	1·23	1·19	1·13
22	1·32	1·32	1·32	1·31	1·29	1·28	1·25	1·22	1·18	1·13
23	1·32	1·32	1·31	1·30	1·29	1·27	1·25	1·22	1·18	1·13
24	1·31	1·30	1·30	1·29	1·28	1·26	1·24	1·21	1·17	1·12
25	1·30	1·30	1·29	1·28	1·27	1·25	1·23	1·21	1·17	1·12
26	1·29	1·29	1·29	1·28	1·27	1·25	1·23	1·20	1·17	1·12
27	1·29	1·28	1·28	1·27	1·26	1·24	1·22	1·20	1·66	1·11
28	1·28	1·28	1·27	1·27	1·25	1·24	1·22	1·19	1·16	1·11
29	1·27	1·27	1·27	1·26	1·25	1·23	1·21	1·19	1·15	1·11
30	1·27	1·27	1·26	1·25	1·24	1·23	1·21	1·18	1·15	1·11

TABLE III

$P = \cdot 95$

N / ρ	0	·1	·2	·3	·4	·5	·6	·7	·8	·9
3	4·35	4·32	4·26	4·15	4·01	3·82	3·57	3·24	2·82	2·23
4	3·04	3·03	2·99	2·93	2·84	2·72	2·56	2·36	2·10	1·75
5	2·52	2·51	2·49	2·44	2·37	2·28	2·16	2·01	1·82	1·56
6	2·24	2·23	2·21	2·17	2·12	2·04	1·95	1·83	1·67	1·46
7	2·06	2·06	2·04	2·01	1·96	1·89	1·81	1·71	1·58	1·40
8	1·94	1·94	1·92	1·89	1·85	1·79	1·72	1·63	1·51	1·35
9	1·85	1·84	1·83	1·80	1·77	1·71	1·65	1·57	1·46	1·32
10	1·79	1·78	1·76	1·75	1·71	1·66	1·61	1·53	1·43	1·30
11	1·72	1·72	1·70	1·68	1·65	1·61	1·55	1·48	1·40	1·28
12	1·67	1·67	1·66	1·64	1·61	1·57	1·52	1·45	1·37	1·26
13	1·63	1·63	1·62	1·60	1·57	1·53	1·49	1·43	1·35	1·24
14	1·60	1·60	1·59	1·57	1·54	1·51	1·46	1·41	1·33	1·23
15	1·57	1·57	1·56	1·54	1·52	1·48	1·44	1·39	1·32	1·22
16	1·55	1·54	1·53	1·52	1·49	1·46	1·42	1·37	1·30	1·21
17	1·52	1·52	1·51	1·49	1·47	1·44	1·40	1·35	1·29	1·20
18	1·50	1·50	1·49	1·48	1·45	1·42	1·39	1·34	1·28	1·20
19	1·49	1·48	1·47	1·46	1·44	1·41	1·37	1·33	1·27	1·19
20	1·47	1·46	1·46	1·44	1·42	1·40	1·36	1·32	1·26	1·18
21	1·45	1·45	1·44	1·43	1·41	1·38	1·35	1·31	1·25	1·18
22	1·44	1·44	1·43	1·42	1·40	1·37	1·34	1·30	1·24	1·17
23	1·43	1·42	1·42	1·40	1·39	1·36	1·33	1·29	1·24	1·17
24	1·41	1·41	1·40	1·39	1·37	1·35	1·32	1·28	1·23	1·16
25	1·40	1·40	1·39	1·38	1·37	1·34	1·31	1·28	1·23	1·16
26	1·39	1·39	1·38	1·37	1·36	1·33	1·31	1·27	1·22	1·16
27	1·38	1·38	1·38	1·36	1·35	1·33	1·30	1·26	1·22	1·15
28	1·38	1·37	1·37	1·36	1·34	1·32	1·29	1·26	1·21	1·95
29	1·37	1·36	1·36	1·35	1·33	1·31	1·28	1·25	1·21	1·15
30	1·36	1·36	1·35	1·34	1·33	1·30	1·28	1·25	1·20	1·14

TABLE IV

$P = \cdot 99$

N / ρ	0	·1	·2	·3	·4	·5	·6	·7	·8	·9
3	9·95	9·87	9·75	9·50	9·13	8·99	8·00	7·17	6·08	4·51
4	5·42	5·39	5·26	5·19	5·00	4·74	4·42	3·99	3·43	2·66
5	3·99	3·97	3·92	3·83	3·70	3·52	3·30	3·00	2·62	2·10
6	3·31	3·29	3·25	3·18	3·08	2·94	2·76	2·54	2·25	1·85
7	2·90	2·90	2·86	2·80	2·72	2·60	2·45	2·27	2·03	1·70
8	2·64	2·63	2·60	2·55	2·48	2·38	2·25	2·09	1·88	1·60
9	2·45	2·44	2·42	2·37	2·30	2·22	2·11	1·96	1·78	1·54
10	2·33	2·32	2·29	2·25	2·19	2·11	2·01	1·88	1·72	1·49
11	2·20	2·19	2·17	2·13	2·07	2·01	1·91	1·80	1·65	1·45
12	2·11	2·10	2·08	2·05	2·00	1·93	1·85	1·74	1·60	1·41
13	2·03	2·03	2·01	1·98	1·93	1·87	1·79	1·69	1·56	1·39
14	1·97	1·97	1·93	1·92	1·87	1·82	1·74	1·65	1·53	1·37
15	1·92	1·91	1·90	1·87	1·83	1·77	1·70	1·61	1·50	1·35
16	1·87	1·87	1·85	1·82	1·79	1·73	1·67	1·58	1·48	1·33
17	1·83	1·83	1·81	1·79	1·75	1·70	1·64	1·56	1·45	1·32
18	1·80	1·79	1·78	1·75	1·72	1·67	1·61	1·53	1·44	1·30
19	1·76	1·76	1·75	1·72	1·69	1·64	1·59	1·51	1·42	1·29
20	1·73	1·73	1·72	1·69	1·66	1·62	1·56	1·49	1·40	1·28
21	1·71	1·70	1·69	1·67	1·64	1·60	1·54	1·48	1·39	1·27
22	1·69	1·68	1·67	1·65	1·62	1·58	1·53	1·46	1·38	1·26
23	1·66	1·66	1·65	1·63	1·60	1·56	1·51	1·45	1·37	1·25
24	1·64	1·64	1·63	1·61	1·58	1·55	1·50	1·43	1·35	1·25
25	1·63	1·62	1·61	1·59	1·56	1·53	1·48	1·42	1·35	1·24
26	1·61	1·61	1·59	1·58	1·55	1·51	1·47	1·41	1·34	1·23
27	1·59	1·59	1·58	1·56	1·54	1·50	1·46	1·40	1·33	1·23
28	1·58	1·58	1·56	1·55	1·52	1·49	1·45	1·39	1·32	1·22
29	1·56	1·56	1·55	1·53	1·51	1·48	1·43	1·38	1·31	1·22
30	1·55	1·55	1·54	1·52	1·50	1·47	1·43	1·37	1·31	1·21

Significant values for the ratio for different values of the correlation coefficient $\rho = 0 \cdot 1$, $0 \cdot 2$, $0 \cdot 3$, $0 \cdot 4$, $0 \cdot 5$, $0 \cdot 6$, $0 \cdot 7$, $0 \cdot 8$, $0 \cdot 9$ and for all values of N between 3 and 30 and for different levels of significance, $P = 0 \cdot 80$, $0 \cdot 90$, $0 \cdot 95$ and $0 \cdot 99$ have been tabulated on the assumption that both the sample standard deviations are measured in identical units, *i.e.*, the null hypothesis $\sigma_1^2 = \sigma_2^2$.

Summary

In this note the distribution of the ratio of sample standard deviations in random samples of size N drawn from a bi-variate correlated normal population has been obtained. The values of this ratio for different values of correlation coefficient (from $0 \cdot 1$ to $0 \cdot 9$) and for N varying from 3 to 30 and for different probability levels, $0 \cdot 8$, $0 \cdot 9$, $0 \cdot 95$ and $0 \cdot 99$ have been tabulated on the assumption of the null hypothesis $\sigma_1^2 = \sigma_2^2$.

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